

Upper bound for the Laplacian eigenvalues of a graph

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Abstract

In this note we give a new upper bound for the Laplacian eigenvalues of an unweighted graph. Let G be a simple graph on n vertices. Let $d_m(G)$ and $\lambda_{m+1}(G)$ be the m -th smallest degree of G and the $m + 1$ -th smallest Laplacian eigenvalue of G respectively. Then $\lambda_{m+1}(G) \leq d_m(G) + m - 1$ for $\bar{G} \neq K_m + (n - m)K_1$. We also introduce upper and lower bound for the Laplacian eigenvalues of weighted graphs, and compare it with the special case of unweighted graphs.

This note is dedicated to Professor Abraham Berman in recognition of his important contributions to linear algebra and for his inspiring guidance and encouragement

1. Introduction

Let $G = (E(G), V(G))$ be a simple graph with $|V(G)| = n$. We say that G is a *weighted graph* if it has a weight (a positive number) associated with each edge. The weight of an edge $\{i, j\} \in E(G)$ will be denoted by w_{ij} . We define the *adjacency matrix* $A(G)$ of G to be a symmetric matrix which satisfies

$$a_{ij} = \begin{cases} 0 & \text{if } \{i, j\} \notin E(G) \\ w_{ij} & \text{if } \{i, j\} \in E(G) \end{cases}$$

The degree diagonal matrix of G will be denoted by $D(G) = \text{diag}(d_1(G), d_2(G), \dots, d_n(G))$ where $d_i(G) = \sum_j a_{ij}$ is the degree of the vertex v_i , and we assume without loss of generality that

$d_1(G) \leq d_2(G) \leq \dots \leq d_n(G)$. We denote by $L(G) = D(G) - A(G)$ the Laplacian matrix of G , and its eigenvalues arranged in increasing order: $0 = \lambda_1(G) \leq \lambda_2(G) \leq \dots \leq \lambda_n(G)$. The largest and the smallest degrees of G will be denoted by $\Delta(G)$ and $\delta(G)$ respectively.

It is well known, e.g. [5], that for a simple unweighted graph G , the eigenvalues of G , and of its complement graph \bar{G} , are related in the following way:

$$(*) \quad \lambda_i(\bar{G}) = n - \lambda_{n-i+2}(G)$$

There are several known lower bounds on the Laplacian eigenvalues of unweighted graph. Grone and Merris [2] proved that $\lambda_n(G) \geq d_n(G) + 1$. Li and Pan [4] showed that $\lambda_{n-1}(G) \geq d_{n-1}(G)$. And finally, Guo[3] gave a lower bound on the third largest Laplacian eigenvalue: $\lambda_{n-2}(G) \geq d_{n-2}(G) - 1$. Brouwer and Haemers generalized these bounds in the following theorem[1]:

Theorem 1. *Let G be finite simple unweighted graph on n vertices. If G is not $K_{n-i+1} + (i-1)K_1$, then $\lambda_i(G) \geq d_i(G) - n + i + 1$.*

2. The unweighted case

The main result in this note is:

Theorem 2. *Let G be finite simple unweighted graph on n vertices. If $\bar{G} \neq K_m + (n - m)K_1$, then $\lambda_{m+1}(G) \leq d_m(G) + m - 1$.*

Proof. The degree sequence of \bar{G} is $n - 1 - d_n(G) \leq n - 1 - d_{n-1}(G) \leq \dots \leq n - 1 - d_1(G)$. From this it follows by Theorems 1 and (*) that:

$$\lambda_{n-m+1}(\bar{G}) = n - \lambda_{m+1}(G) \quad (1)$$

$$\lambda_{n-m+1}(\bar{G}) \geq d_{n-m+1}(\bar{G}) - m + 2 \text{ for } \bar{G} \neq K_m + (n-m)K_1 \quad (2)$$

$$d_{n-m+1}(\bar{G}) = n - 1 - d_m(G) \quad (3)$$

From (1) and (2) we get $n - \lambda_{m+1}(G) \geq d_{n-m+1}(\bar{G}) - m + 2$. Combining it with (3) yields:
 $n - \lambda_{m+1}(G) \geq n - 1 - d_m(G) - m + 2$. Therefore $\lambda_{m+1}(G) \leq d_m(G) + m - 1$ for $\bar{G} \neq K_m + (n-m)K_1$. \square

One of the consequences of this theorem, is that now we can describe the intervals in which the Laplacian eigenvalues lies.

Corollary 1. *Let G be finite simple unweighted graph on n vertices, and suppose $G \neq K_{n-m+1} + (m-1)K_1$ and $\bar{G} \neq K_{m-1} + (n-m+1)K_1$. Then $d_m(G) - n + m + 1 \leq \lambda_m(G) \leq d_{m-1}(G) + m - 2$. In addition, if $d_m(G) = d_{m-1}(G) + n - 3$, then the inequality in Theorems 1 and 2 becomes equality.*

Example : Let G be a graph that is constructed from $K_{1,n-1}$ by adding an edge (or edges) between pairs of unconnected vertices of $K_{1,n-1}$, such that $d_{n-1}(G) = 2$. Then equality holds for $\lambda_n(G)$.

Proof. The first part follows immediately from Theorems 1 and 2. Now, if $d_m(G) = d_{m-1}(G) + n - 3$, then $d_m(G) - n + m + 1 = d_{m-1}(G) + n - 3 - n + m + 1 = d_{m-1}(G) + m - 2$, hence the inequalities becomes equalities. \square

3. The weighted case

Let G be a simple weighted graph. Let $l_m = \{v_m, v_{m+1}, \dots, v_n\}$ be the set of $n - m + 1$ vertices with the largest degrees in G , and let $S_m = \{v_1, v_2, \dots, v_m\}$ be the set of m vertices with the smallest degrees in G . Define G_{l_m} and G_{S_m} be the subgraphs induced from G by l_m and S_m respectively. Now we are ready to prove the following:

Theorem 3. *Let G be a simple weighted graph. Then $\lambda_m(G) \geq d_m(G) - \Delta(G_{l_m})$*

Proof. We will use the following well known result about eigenvalues, e.g [6].

(**) Let A be a symmetric matrix. Then $\lambda_k(A) = \max\left\{\frac{\langle Af, f \rangle}{\langle f, f \rangle} \mid f \perp f_{k+1}, f_{k+2}, \dots, f_n\right\}$, when $f_{k+1}, f_{k+2}, \dots, f_n$ are eigenvectors of the eigenvalues $\lambda_{k+1}, \lambda_{k+2}, \dots, \lambda_n$ respectively.

Let $f_{m+1}, f_{m+2}, \dots, f_n$ be the eigenvectors of the eigenvalues $\lambda_{m+1}, \lambda_{m+2}, \dots, \lambda_n$ respectively. We define a vector g in the following way:

$$g_j = \begin{cases} 0 & \text{if } j \notin l_m \\ c_{v_j} & \text{if } v_j \in l_m \end{cases}$$

,where $\{c_{v_m}, c_{v_{m+1}}, \dots, c_{v_n}\}$ is a set of variables such that $g \perp f_i$ for all $m+1 \leq i \leq n$. Note that since we have $n - m$ equations, with $n - m + 1$ variables, there exists such nontrivial set (i.e. not all the variables are zero). From (**) it follows:

$$\begin{aligned} \lambda_m(G) &= \max\left\{\frac{\langle Lf, f \rangle}{\langle f, f \rangle} \mid f \perp f_{m+1}, f_{m+2}, \dots, f_n\right\} \geq \frac{\langle L(G)g, g \rangle}{\langle g, g \rangle} \\ &= \frac{\langle D(G)g, g \rangle}{\langle g, g \rangle} - \frac{\langle A(G)g, g \rangle}{\langle g, g \rangle} \\ &= \frac{\sum_{j=m}^n d_j(G) c_{v_j}^2}{\sum_{j=m}^n c_{v_j}^2} - \frac{\langle A(G)g, g \rangle}{\langle g, g \rangle} \end{aligned} \quad (4)$$

Now, from (**) and the construction of g , $\frac{\langle A(G)g, g \rangle}{\langle g, g \rangle}$ is smaller than or equal to the largest eigenvalue of $A(G_{l_m})$, which, according to the Gersgorin Disc Theorem (e.g [6]), is smaller than or equal to $\Delta(G_{l_m})$. Finally, from (4) we get $\lambda_m(G) \geq d_m(G) - \Delta(G_{l_m})$ \square

Our goal now is to find an upper bound on the eigenvalues of weighted graph. Let a be the maximal weight of an edge in G . We normalize G by dividing its weights by a , and denote the normalized graph by \hat{G} . The complement graph of \hat{G} , $\tilde{\hat{G}}$ is defined in the following way: $V(\tilde{\hat{G}}) = V(\hat{G})$, $w_{ij}(\tilde{\hat{G}}) = 1 - w_{ij}(\hat{G})$. For \hat{G} and $\tilde{\hat{G}}$, (*) holds (e.g. [5]).

We are ready now to present the upper bound for the Laplacian eigenvalues of weighted graphs:

Theorem 4. *Let G be a simple weighted graph, and let a be the maximal weight of an edge in G . Then $d_m(G) + ma - \delta(G_{S_m}) \geq \lambda_{m+1}(G)$.*

Proof. The proof is very similar to the proof of theorem 2. First, we define the graph \hat{G} , which is constructed from G as we explained above. We notice that S_m is the set of m vertices with the largest degrees in \hat{G} . The graph H would be the graph that induced from \hat{G} by S_m . Propositions (1), (3) (from the proof of Theorem 2) still holds for \hat{G} , and instead of (2), we can write, using Theorem 3,

$$\lambda_{n-m+1}(\tilde{\hat{G}}) \geq d_{n-m+1}(\tilde{\hat{G}}) - \Delta(H).$$

By following the steps from the proof of Theorem 2, we get

$$d_m(\hat{G}) + 1 + \Delta(H) \geq \lambda_{m+1}(\hat{G}).$$

and since

$$\Delta(H) = m - 1 - \delta(G_{S_m})$$

we have

$$d_m(\hat{G}) + m - \delta(G_{S_m}) \geq \lambda_{m+1}(\hat{G}).$$

We conclude that

$$d_m(G) + ma - \delta(G_{S_m}) \geq \lambda_{m+1}(G)$$

\square

Corollary 2. *Let G be finite simple weighted graph on n vertices, and denote by a the maximal weight of an edge in G . Then $d_m(G) - \Delta(G_{l_m}) \leq \lambda_m(G) \leq d_{m-1}(G) + (m-1)a - \delta(G_{S_{m-1}})$.*

In the special case of unweighted graphs, the bounds reduce to $d_m(G) - n + m \leq \lambda_m(G) \leq d_{m-1}(G) + m - 1$. It is interesting to compare it with the stronger result from Section 2: $d_m(G) - n + m + 1 \leq \lambda_m(G) \leq d_{m-1}(G) + m - 2$.

References

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